Regular and anomalous scaling of a randomly advected passive scalar

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Extending Kolmogorov's refined similarity hypothesis to study the inertial behavior $\langle [T(\mathbf{x}+\mathbf{r},t) - T(\mathbf{x},t)]^{2n}\rangle \propto r^{\zeta_{2n}}$ of a passive scalar $T(\mathbf{x},t)$ advected by a rapidly changing incompressible velocity field, a random variable θ was introduced by Ching [Phys. Rev. Lett. **79**, 3644 (1997)]. In this paper, the statistical distribution of the random variable $X=\theta/\sqrt{\langle\theta^2\rangle}$ is investigated analytically for the scaling in two limits, n-independent scaling $\zeta_{2n}=\zeta_2$ and regular scaling $\zeta_{2n}=n\zeta_2$, and numerically for the scaling of the Kraichnan conjecture $\zeta_{2n}=\frac{1}{2}[\sqrt{4nd\zeta_2+(d-\zeta_2)^2}-(d-\zeta_2)]$. For n-independent scaling $\zeta_{2n}=\zeta_2$, the statistical distribution of X tends to an exponential distribution when $\zeta_2\rightarrow 0$ or $d\rightarrow\infty$ and to a Gaussian distribution when $\zeta_2 \rightarrow 2$ and $d=2$. For regular scaling $\zeta_{2n}=n\zeta_2$, the statistical distribution of X tends to a Gaussian distribution when $\zeta_2\rightarrow 0$ or $d\rightarrow\infty$. In $d=2$, there seems to be a phase transition for the probability density function $P(X)$ from a convex to a concave function when the value of ζ_2 is increased and the critical point is $\zeta_2=4/3$ where the random variable X has a uniform distribution in $[-\sqrt{3},\sqrt{3}]$. In $d=3$, $P(X)$ is a convex function for all $0<\zeta_2<2$ and tends to a constant on its support $[-\sqrt{3},\sqrt{3}]$ when $\zeta_2\rightarrow 2$. For the scaling of the Kraichnan conjecture, $P(X)$ has two peaks in $d=2$ for $\zeta_2>1.33$, but, in $d=3$, it has only one peak for all $0<\zeta_2<2$ and changes very slowly with the value of X in the neighborhood of $X=0$ as $\zeta_2\rightarrow 2$.

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The problem of passive scalar advection is of physical importance in itself and may also serve as starting point in studying multiscaling in turbulence. Although the governing equation for a passive scalar is apparently linear when a random velocity field is prescribed, the statistical properties of the passive scalar are much more elusive. Nevertheless, a mathematically tractable model in which the velocity field is rapidly changing in time was introduced by Kraichnan [1]. Some results of this model can be obtained exactly. Based upon a linear ansatz [2], Kraichnan gave a conjecture for the scaling exponents ζ_{2n} of the scalar structure functions $S_{2n}(r)$ defined by $S_{2n}(r)\sim r^{\zeta_{2n}}$, where $S_{2n}(r)\equiv\langle [T(\mathbf{x}+\mathbf{r},t) - T(\mathbf{x},t)]^{2n}\rangle \equiv \langle [T_r(\mathbf{x})]^{2n}\rangle$. The simplified model has attracted much attention as a possible model to study intermittency and multiscaling in turbulence [2–9]. Kolmogorov's refined similarity hypothesis (RSH) [10,11] was extended to study the intermittency of this model by Ching [12]. With the RSH formulated in terms of a random variable θ, the molecular-diffusion terms were evaluated. The scaling exponents ζ_{2n} of the scalar structure functions $S_{2n}(r)$ are determined solely by the statistics of the random variable θ. Therefore, the relation between the scaling exponents ζ_{2n} and the statistical distribution of θ is of great interest.

The Hölder reliability inequalities require that $d\zeta_{2n}/dn$ be a nonincreasing function of n. This implies that $\zeta_{2n} \leqslant n\zeta_2$. The limit $\zeta_{2n}=n\zeta_2$ is called regular scaling. Away from this limit, the scaling is called anomalous scaling. On the other hand, all the theoretical analysis and numerical simulations suggest that ζ_{2n} is a nondecreasing function of n [2,3,8,9]. This requires $\zeta_{2n}\geqslant\zeta_2$. The physical picture for the limit $\zeta_{2n}=\zeta_2$ was given in [2]. The Kraichnan conjecture based upon the linear ansatz suggests that the scaling is anomalous midway between the two limits $\zeta_{2n}=\zeta_2$ and $\zeta_{2n}=n\zeta_2$. The scaling exponent ζ_2 is taken to be in the region $0<\zeta_2<2$ [2]. In this paper, we investigate the statistical properties of θ for the scaling in the two limits of n-independent scaling $\zeta_{2n}=\zeta_2$ and regular scaling $\zeta_{2n}=n\zeta_2$, and also for the scaling of Kraichnan conjecture.

Consider the structure functions $S_{2n}(r)=\langle [T_r(\mathbf{x})]^{2n}\rangle$ of a passive scalar field $T(\mathbf{x},t)$ that obeys

$$\frac{\partial T}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})T=\kappa\nabla^2 T. \tag{1}$$

In the limit of an infinitely rapid change in time of $\mathbf{v}(\mathbf{x},t)$, the equation for $S_{2n}(r)$ is given by [1,2,9]

$$-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\eta(r)\frac{\partial S_{2n}(r)}{\partial r}\right)=J_{2n}(r), \tag{2}$$

where d is the space dimensionality and $\eta(r)$ is the two-particle eddy-diffusivity scalar with the form $\eta(r) = \eta(L)(r/L)^{(2-\zeta_2)}, L$ denoting the integral scale. The molecular-diffusion terms $J_{2n}(r)$ in Eq. (2) are

$$J_{2n}(r)=2n\kappa\langle T_r^{2n-1}(\nabla_x^2+\nabla_{x'}^2)T_r\rangle. \tag{3}$$

The existence of a power law for $S_{2n}(r)$ requires [2,9]

$$J_{2n}(r)=nC_{2n}J_2\frac{S_{2n}(r)}{S_2(r)}, \tag{4}$$

where C_{2n} are dimensionless constants that must be determined. Equation (4) is also a consequence of the fusion rule proposed by L'vov and Procaccia [13–15]. From Eqs. (2) and (4), the scaling exponents ζ_{2n} can be evaluated [2]:

$$\zeta_{2n}=\frac{1}{2}[\sqrt{4nC_{2n}d\zeta_2+(d-\zeta_2)^2}-(d-\zeta_2)]. \tag{5}$$

Based upon the linear ansatz $\langle(\nabla_x^2+\nabla_{x'}^2)T_r|T_r\rangle\propto T_r$, the closure $C_{2n}=1$ was proposed by Kraichnan [2]. The scaling exponents ζ_{2n} of the Kraichnan conjecture are $\zeta_{2n}=\frac{1}{2}[\sqrt{4nd\zeta_2+(d-\zeta_2)^2}-(d-\zeta_2)]$.

Numerical simulations and experimental data [3,9,15] indicate that the linear ansatz may be approximately correct, but a small departure from the linear ansatz (especially for small values of T_r) can create large changes in the anomalous exponents. Therefore, there have been many studies of the scaling of a passive scalar using alternative methods [4–8]. Ching formulated the RSH as

$$T_r^2=\theta^2\frac{L^2}{\eta(L)}\left(\frac{r}{L}\right)^{\zeta_2}\chi_r, \tag{6}$$

where χ_r is the locally averaged scalar dissipation rate defined by $\chi_r=(1/V_r)\int_{B_r}\kappa|\nabla T(\mathbf{y})|^2d\mathbf{y}$ with B_r being a d-dimensional ball centered at $\mathbf{x}$ with radius r and volume V_r and θ a dimensionless random variable independent of r and statistically independent of $\chi(r)$. Using the formulation (6), Ching [12] evaluated the molecular-diffusion terms $J_{2n}(r)$ and showed that the scaling exponents ζ_{2n} satisfy

$$\zeta_{2n}=\frac{1}{2}\{\sqrt{[d-\zeta_2-g(n)\zeta_2]^2+4ng(n)\zeta_2(d-\zeta_2)}-[d-\zeta_2-g(n)\zeta_2]\}, \tag{7}$$

where $g(n)=(2n-1)\langle X^{2n-2}\rangle/\langle X^{2n}\rangle$, in which $X=\theta/\sqrt{\langle\theta^2\rangle}$. From Eq. (7), we can find all the integer moments of the random variable X if the scaling exponents ζ_{2n} are already known.

The statistical distribution of X can be determined uniquely from its integer-moments $\langle X^n\rangle$ if the power series $\Sigma_{n=1}^{\infty}[\langle X^{2n}\rangle\lambda^n/(2n)!]$ converges in some interval $|\lambda|<\lambda_0$, where $\lambda_0>0$ is the radius of convergence [16]. This sufficient condition is equivalent to $\lim_{n\to\infty}\langle X^{2n}\rangle/[2n(2n-1)\langle X^{2n-2}\rangle]=\lim_{n\to\infty}1/[2ng(n)]<1/\lambda_0<\infty$. From Eq. (7), $\zeta_{2n}\geqslant\zeta_2$ leads to $g(n)\geqslant d/[nd+(1-n)\zeta_2]$. Thus, for the scaling of a passive scalar, the sufficient condition is satisfied naturally, so that the statistical distribution of X can be determined uniquely from its moments.

In the following, we give analytical solutions for the probability density function (PDF) $P(X)$ for the scaling at the two limits of n-independent scaling $\zeta_{2n}=\zeta_2$ and regular scaling $\zeta_{2n}=n\zeta_2$, and numerical calculations for the scaling of the Kraichnan conjecture.

For n-independent scaling $\zeta_{2n}=\zeta_2$, from $g(n)=d/[nd+(1-n)\zeta_2]$, we have

$$\int X^{2n}P(X)dX=\int(2n-1)\times\left[\left(1-\frac{\zeta_2}{d}\right)n+\frac{\zeta_2}{d}\right]X^{2n-2}P(X)dX. \tag{8}$$

Using the method of Sinai and Yakhot [17], from Eq. (8) we can obtain

$$\int\left[\frac{1}{2}\left(1-\frac{\zeta_2}{d}\right)X\frac{d^2P}{dX^2}-\frac{\zeta_2}{d}\frac{dP}{dX}-XP\right]X^{2n-1}=0. \tag{9}$$

Since $P(X)$ is an even function of X, we have

$$\frac{d^2P}{dX^2}-\frac{2\nu}{X}\frac{dP}{dX}-c^2P=0, \tag{10}$$

where $\nu=\zeta_2/(d-\zeta_2)$ and $c=\sqrt{2d/(d-\zeta_2)}$. Replacing X and $P(X)$ by $z=cX$ and $P=(c^{-1}z)^{\nu+1/2}F(z)$, we can turn Eq. (10) into the Bessel modified equation $d^2F/dz^2+(1/z)dF/dz-[1+(\nu+1/2)^2/z^2]F=0$. Noting that $\lim_{x\to\pm\infty}P(X)=0$, we have $P(X)=A|X|^{\nu+1/2}K_{\nu+1/2}(c|X|)$ where $K_{\nu+1/2}(c|X|)$ is a modified Bessel function of the second kind of order $\nu+1/2$ and A is a constant. Determining the constant A from the condition $\int_{-\infty}^{+\infty}P(X)=1$, the solution for Eq. (10) is

$$\begin{aligned}P(X)&=\frac{c^{\nu+3/2}}{2^{\nu+1/2}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\right)}K_{\nu+\frac{1}{2}}(c|X|)\\&=\frac{c^{2\nu+2}}{\pi}|X|^{2\nu+1}\int_0^{\infty}\frac{\cos(t)}{(t^2+c^2X^2)^{\nu+1}}dt\\&=\frac{c}{\pi}\int_0^{\infty}\frac{\cos(c|X|t)}{(1+t^2)^{\nu+1}}dt.\end{aligned} \tag{11}$$

From the analytical solution (11), we can obtain the following results.

(1) For $\zeta_2\to0$ or $d\to\infty$, we have $\nu\to0$. Thus, the statistical distribution of X tends to the exponential distribution

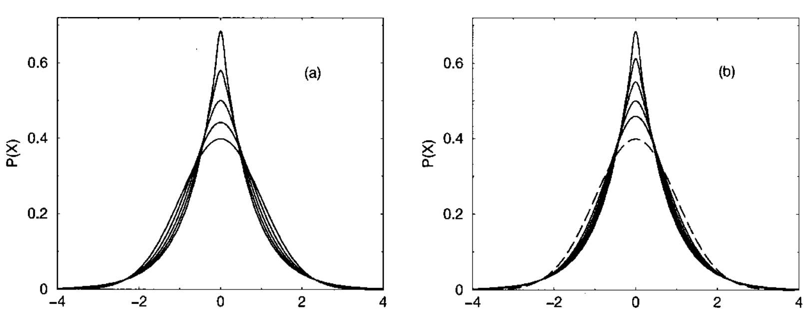

FIG. 1. Plot of the probability density function $P(X)$ as a function of $X=\theta/\sqrt{\langle\theta^2\rangle}$ for n-independent scaling $\zeta_{2n}=\zeta_2$. (a) $d=2$ for different $\zeta_2=0.0$ (the exponential distribution), 0.5, 1.0, 1.5, 2.0 (the Gaussian distribution) (from top to bottom), and (b) $d=3$ for different $\zeta_2=0.0$ (the exponential distribution), 0.5, 1.0, 1.5, 2.0, and the Gaussion distribution (dashed line) (from top to bottom).

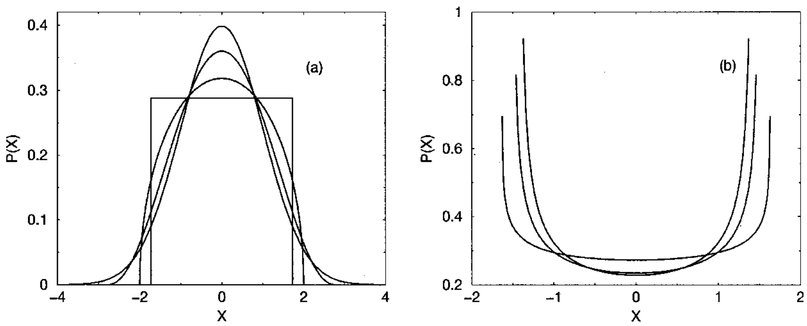

FIG. 2. Plot of the probability density function $P(X)$ as a function of $X=\theta/\sqrt{\langle\theta^2\rangle}$ for regular scaling $\zeta_{2n}=n\zeta_2$ in $d=2$ for (a) $\zeta_2=0.0$ (the Gaussian distribution), 0.5, 1.0, 4/3 (the uniform distribution in $[-\sqrt{3},\sqrt{3}]$) (from top to bottom) and (b) $\zeta_2=1.5,1.8,2.0$ (from bottom to top).

$$\lim_{\zeta_2\to 0, d\to\infty} P(X)=\frac{1}{\sqrt{2}}e^{-\sqrt{2}X}.$$

(2) In $d=2$, the functions $g(n)$ tend to 1 when $\zeta_2\to 2$. In this case, the statistical distribution of X will tend to the Gaussian distribution

$$\lim_{\zeta_2\to 2} P(X,d=2)=\frac{1}{\sqrt{2\pi}}e^{-X^2/2}.$$

Figures 1(a) and 1(b) show the probability density function $P(X)$ in $d=2$ and $d=3$ for different ζ_2.

For regular scaling $\zeta_{2n}=n\zeta_2$, the relation $g(n)=n\zeta_2/d+(1-\zeta_2/d)$ gives

$$\int\left[\frac{\zeta_2}{d}n+\left(1-\frac{\zeta_2}{d}\right)\right]X^{2n}P(X)dX = \int(2n-1)X^{2n-2}P(X)dX. \quad (12)$$

This implies

$$\left(1-\frac{\zeta_2}{2d}X^2\right)\frac{dP}{dX}=\left(\frac{3\zeta_2}{2d}-1\right)XP. \quad (13)$$

Since $P(X)$ is a non-negative function with the condition $\int_{-\infty}^{+\infty}P(X)=1$, $P(X)$ is concentrated in a finite interval $[-\sqrt{2d/\zeta_2},\sqrt{2d/\zeta_2}]$:

$$P(X)=B\left(1-\frac{\zeta_2}{2d}X^2\right)^{d/\zeta_2-3/2}, \quad (14)$$

where $B=[\sqrt{2d/\zeta_2}\Gamma(1/2)\Gamma(d/\zeta_2-1/2)/\Gamma(d/\zeta_2)]^{-1}$. So we have the following results.

(1) For $\zeta_2\to 0$ or $d\to\infty$, $g(n)\to 1$. The statistical distribution of X tends to a Gaussian distribution.

(2) In $d=2$, when $\zeta_2=4/3$, the random variable X has a uniform distribution in $[-\sqrt{3},\sqrt{3}]$. It is interesting that the probability density function $P(X)$ is a convex function for $\zeta_2<4/3$ and a concave function for $\zeta_2>4/3$ [see Figs. 2(a) and 2(b)]. This appears to be a phase transition and the critical point is $\zeta_2=4/3$. In $d=3$, the situation is different. The probability density function $P(X)$ is a convex function for all $0<\zeta_2<2$ and tends to a constant on its support $[-\sqrt{3},\sqrt{3}]$ when $\zeta_2\to 2$ [see Fig. 3].

It is surprising that similar distribution to (14) has been found in the investigation of the RSH of a turbulent velocity field [11] and for the statistical quasistationary PDF of the passive scalar $X=(T-\langle T\rangle)/\langle(T-\langle T\rangle)^2\rangle^{1/2}$ in turbulence [18].

For the scaling of the Kraichnan conjecture $\zeta_{2n}=\frac{1}{2}[\sqrt{4nd\zeta_2+(d-\zeta_2)^2}-(d-\zeta_2)]$, we can calculate the moments $\langle X^{2n}\rangle$ for all positive integers n from $g(n)=1.0/(1.0-\zeta_2/d+\zeta_{2n}/nd)$. The characteristic function $f(t)$ of the probability density function $P(X)$ is

$$f(t)=1+\sum_{n=1}^{\infty}\frac{(-1)^n\langle X^{2n}\rangle}{(2n)!}t^{2n}.$$

Noting that $f(t)$ is an even function of t, we have

$$P(X)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(t)e^{-ixt}dt=\frac{1}{\pi}\int_0^{+\infty}f(t)\cos(xt)dt.$$

Using the technique for numerical integration of an oscillating function [19], we obtain the numerical results for the probability density function $P(X)$ in $d=2$ and $d=3$ [see Figs. 4(a) and 4(b)]. Based upon the numerical calculations, we have the following results.

(1) For $\zeta_2\to 0$ or $d\to\infty$, the statistical distribution of X tends to the Gaussian distribution.

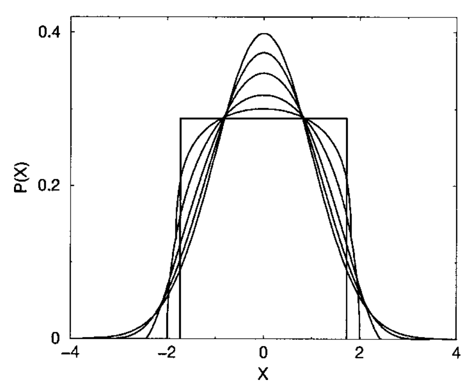

FIG. 3. Plot of the probability density function $P(X)$ as a function of $X=\theta/\sqrt{\langle\theta^2\rangle}$ for regular scaling $\zeta_{2n}=n\zeta_2$ in $d=3$ for different $\zeta_2=0.0,0.5,1.0,1.5,1.8,2.0$ (from top to bottom) where $\zeta_2=0.0$ and 2.0 correspond to the Gaussian distribution and the uniform distribution in $[-\sqrt{3},\sqrt{3}]$, respectively.

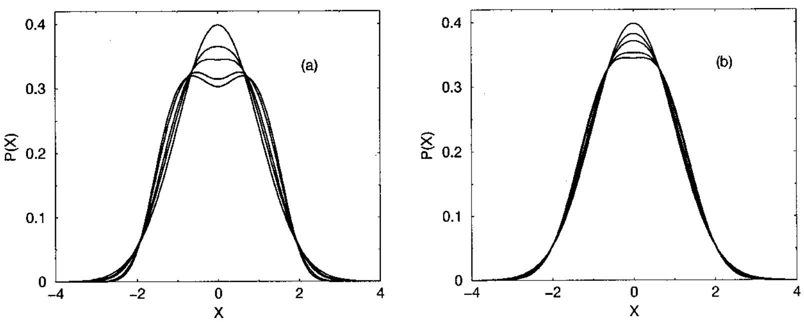

FIG. 4. Plot of the probability density function $P(X)$ as a function of $X=\theta/\sqrt{\langle\theta^2\rangle}$ for the Kraichnan conjecture for different $\zeta_2=0.0$ (the Gaussian distribution), 1.0, 4/3, 1.8, 2.0 (from top to bottom) in (a) $d=2$ and (b) $d=3$.

(2) It is interesting that the probability density function $P(X)$ has a saddle shape with two peaks in $d=2$ when $\zeta_2>1.33$. In $d=3$, it has only one peak at $X=0$ for all $0<\zeta_2<2$ and changes very slowly with the value of X in the neighborhood of $X=0$ when $\zeta_2\to 2$.

Therefore, we suggest that the statistical distribution of X for the scaling of the Kraichnan conjecture has some similar properties to the regular scaling $\zeta_{2n}=n\zeta_2$ in the neighborhood of $X=0$ and also some similar properties to n-independent scaling $\zeta_{2n}=\zeta_2$, e.g., $P(X)$ seems to be concentrated in an infinite interval $(-\infty,\infty)$.

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